Dielectric breakdown model at small η: Pole dynamics

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We consider the dielectric breakdown model in the limit $\eta\to 0^+$. This is shown to lead to Sivashinsky's equation. We show that a particular configuration of poles is linearly stable, in analogy to the stability of the 1/2 finger for diffusion limited aggregation, and compute exactly the eigenvalues of the stability matrix.

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The problem of Laplacian growth in two dimensions has been much studied since the introduction of the model of diffusion limited aggregation (DLA) [1]. The model is known to give rise to complex, branching structures, with a nontrivial fractal dimension. A generalization of DLA is the dielectric breakdown model (DBM) [3], which includes an additional free parameter η, and has a continuously varying fractal dimension. Recently, a renormalzation group approach has been developed based on expanding about $\eta=4$, where it is argued that the clusters become asymptotically one dimensional [2].

An alternative possibility, which will be explored in this paper, is to expand near $\eta=0$. For small η, the cluster becomes less branched, with a dimensionality approaching 2 in the limit $\eta\to 0$, where the dynamics reduces to an Eden model [4]. The idea behind the present work is to examine the DBM in the limit of infinitesimal positive η. We show that in this case, the dynamics leads to Sivashinsky's equation [5]. The pole dynamics is very similar to the branched growth dynamics of Halsey and co-workers [6]. We then consider the pole solutions of this equation, and show that the solution with the maximal number of poles is linearly stable to all perturbations. This is analogous to the stability of the 1/2 finger in DLA [7]. However, we are able to analytically calculate all eigenvalues and eigenvectors of the stability matrix.

For any positive η, a flat surface is unstable under DBM growth. However, in the limit $\eta\to 0^+$, the fluctuations about the flat surface become small. We now show how to rescale the fluctuations in the surface to obtain a nontrivial growth equation describing the $\eta\to 0^+$ limit.

DIFFERENTIAL EQUATION FOR SURFACE GROWTH

First we introduce the continuum equation of motion for the DBM, and then we take the appropriate limit. We rescale the fluctuations in the height of the surface in this limit. We regularize the equation on a short distance, and argue that this regularization corresponds to a finite walker size in the DBM. Finally, we discuss some basic scaling properties of the resulting equation and discuss how to include noise in the dynamics. The final result will be Sivashinsky's equation, with noise.

We consider growth in a strip geometry throughout. Here, the growth is confined to a strip, which is periodic in the horizontal direction, with the growth happening vertically. We will use complex variables to describe the coordinates in the two-dimensional strip in which growth occurs. The real part of the complex variable will represent the horizontal position, while the imaginary part represents the vertical position. Growth will occur in the positive imaginary direction. For the strip boundary conditions, we will use a coordinate x, periodic with period 2π, to parametrize the cluster. The initial, seed cluster is taken to be a flat surface, and random walkers are released from infinity, above the strip. In the strip geometry, one finds fractal DBM clusters when the ratio of the width of the strip to the walker size diverges.

We will parametrize the boundary of the cluster by using a function $F(x)$, such that $F(x)$ yields the cluster boundary, for $x=0,\ldots,2\pi$. $F(x)$ is taken to be analytic, and one to one for x with positive imaginary part. Thus, $F(x)=x+\Sigma_{j=0}^{\infty}F(j)e^{ijx}$. Throughout, we use the symbols $j,k,\ldots$ to refer to Fourier modes, while x will be used to refer to real space. $F(j)=0$ for $j<0$.

The surface does not contain overhangs in our limit, which permits us to describe the surface by its height $h(x)=\mathrm{Im}[F(x)]+\cdots$.

At given η, the dynamics of the cluster can be obtained from the Shraiman-Bensimon equation [8]:

$$\partial_t F(x,t)=i[\partial_x F(x,t)]\int dx'|\partial_{x'}F(x')|^{-1-\eta}\frac{e^{ix'}+e^{ix}}{e^{ix'}-e^{ix}}. \tag{1}$$

Let us write this equation in terms of $F(j)$ and expand this equation in powers of η and a_k. The linear term is

$$\partial_t F(j)=\eta k F(k)-\nu k^2 F(j). \tag{2}$$

We have added an additional regularization to the linearized equation, with coefficient ν proportional to the walker size. Equation (1) needs some such regularization to remove finite time singularities.

Unlike the linear term, the nonlinear terms in Eq. (1) are nonvanishing in the limit $\eta\to 0$. The first nonlinear term is, at $\eta=0$,

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$$\partial_t F(j) = -\sum_{k>0} [k(j-k)F(k)F(j-k) - 2k(j+k)\overline{F}(k)F(j+k)]. \tag{3}$$

We drop the constant term in the expansion of Eq. (1), which simply describes an overall upward motion of the surface.

For small η, $F(k)$ will be of order η, by scaling, so that all higher order nonlinearities will be unimportant in the limit. Combining Eqs. (2) and (3) and rescaling the field F by η, and also rescaling the time coordinate and ν, the equation of motion for the system is

$$\partial_t F(j) = kF(k) - \nu k^2 F(j) - \sum_{k>0} [k(j-k)F(k)F(j-k) - 2k(j+k)\overline{F}(k)F(j+k)]. \tag{4}$$

We have numerically studied the small-η behavior of DBM clusters for small η and verified the scaling $F \propto \eta$. To obtain this scaling, one must take the small-η limit before the limit of vanishing walker size. In the opposite limit, with the walker size taken to zero before η is taken to zero, the fluctuations in surface height remain large. The crossover between these limits remains open.

The differential equation for the growth of the height is

$$\partial_t h(x,t) = |\partial_x| h(x,t) + \nu \partial_x^2 h(x,t) + [\partial_x h(x,t)]^2 + \text{noise}, \tag{5}$$

equivalent to Sivashinky's equation. We have added a random noise field due to shot noise in the number of arriving walkers. The noise will be short-range correlated on a length and time scale set by the short-distance regularization. Thus, the magnitude of the noise will be proportional to ν, as for smaller walkers the shot noise is reduced. The mean-square fluctuations in the noise will be of order ν^2.

POLE DYNAMICS AT NONVANISHING ν

It has been shown [9] that Eq. (4) has pole solutions. Consider the ansatz

$$F(k) = -\sum_{a=1}^{N} \frac{\nu}{k} e^{-i\epsilon_a k}. \tag{6}$$

The points ϵ_a are poles of $\partial_x F(x,t)$ in the complex plane. In this case, the equation of motion yields

$$\partial_t \epsilon_a = i - i\nu - 2i\nu \times \left[\frac{e^{2\,\mathrm{Im}(\epsilon_a)}}{1-e^{2\,\mathrm{Im}(\epsilon_a)}} + \sum_{b\neq a} \left(\frac{1}{1-e^{i\epsilon_b - i\epsilon_a}} + \frac{1}{e^{i\epsilon_a - i\overline{\epsilon}_b}-1} \right) \right], \tag{7}$$

so that $F(k)$ continues to be described by the ansatz (6), and the dynamics of Eq. (7) causes poles to attract in the real direction while repelling in the imaginary direction.

Now consider the stationary states of the pole dynamics (7). With x periodic, there are many possible stationary states. For example, a state with some having real coordinates 0 and some having real coordinates π may be stationary. However, such a state will not be stable under small translations of the pole positions. For stability, all poles must have the same real coordinate. In this case, there is only one possible set of imaginary coordinates that leads to a stationary state. Such a state is stable under any perturbation of pole coordinates, except for the trivial zero mode associated with translation.

For strip boundary conditions there is a maximum number of poles one can use to form these stable configurations. Consider what Eq. (7) implies for such a configuration. Clearly, $\mathrm{Re}(\partial_t \epsilon_a) = 0$ for all a. We also need $\mathrm{Im}(\partial_t \epsilon_a) = 0$ for all a. Let us order the poles so that if $a < b$ then $\mathrm{Im}(\epsilon_a) > \mathrm{Im}(\epsilon_b)$. Consider the equation of motion for ϵ_N. We find that

$$\partial_t \epsilon_N = i - i\nu - i2\nu \times \left[\frac{e^{2\,\mathrm{Im}(\epsilon_N)}}{1-e^{2\,\mathrm{Im}(\epsilon_N)}} + \sum_{b\neq a} \left(\frac{1}{1-e^{i\epsilon_b - i\epsilon_N}} + \frac{1}{e^{i\epsilon_N - i\overline{\epsilon}_b}-1} \right) \right]. \tag{8}$$

The first term on the right-hand side is positive imaginary, while all the other terms are negative imaginary. So long as $\mathrm{Im}(\epsilon_N) < \mathrm{Im}(\epsilon_a)$ for all $a \neq N$, then the imaginary part of the right-hand side is monotonically increasing as $\mathrm{Im}(\epsilon_N)$ becomes more and more negative. In the limit $\epsilon_N \to -i\infty$, the right-hand side becomes

$$i[1-\nu-2\nu(N-1)]. \tag{9}$$

If this quantity is negative imaginary, then it is not possible, for any ϵ_N, to have $\partial_t \epsilon_N = 0$. Then, $\partial_t \mathrm{Im}(\epsilon_N) < 0$ for all time, and we find that ϵ_N moves off to $-i\infty$. In the limit that ϵ_N moves off to infinity, this pole no longer contributes to the sum in Eq. (6) and can be ignored.

Thus, for any ν in the strip geometry, there are states with N poles, stable against any small perturbation in the pole position. For each N, there is only one such state, and it has all poles with the same real coordinate. Such a stable solution with N poles can be constructed only if $-1+\nu+2\nu(N-1)<0$. Otherwise, some number of poles move off to infinity until the solution reduces to one with $-1+\nu+2\nu(N-1)<0$.

STABILITY ANALYSIS

We now consider the linear stability of the stationary states above. Here, we will consider arbitrary small perturbations of h, including perturbations that cannot be written in terms of a small change in the pole coordinates. We are able to analytically calculate the eigenvalues of the linear problem, and find that, for each value of viscosity, only one of the stationary states is linearly stable.

Before doing any mathematical analysis of our problem, recall the stability analysis of the Saffman-Taylor system

[10]. Saffman and Taylor found a one-parameter family of stationary solutions to the zero surface-tension problem. Surface tension acts as a singular perturbation to this problem. Eventually it was shown [7] that a small amount of surface tension stabilizes the 1/2 finger against small perturbations. However, in the DLA problem, although the system is regularized due to nonvanishing walker size, a fractal structure emerges instead of stable fingers. In our system, although the system is stable, in the limit of vanishing walker size an exponentially small (in $1/\nu$) amount of noise will destabilize it.

The physical reason for stability is most easily understood in a WKB approximation. Consider a localized, short-wavelength, perturbation $f(x)$ to a stationary state $F_0(x)$. Let the stationary state be such that all poles have real coordinate 0. Then the surface has a deep depression near $x=0$, and a very broad finger located around $x=\pi$. The perturbation will move in position, due to the term in the equation of motion $\partial_x f(x)\partial_x F_0(x)$, until the perturbation moves to $x=0$, where it disappears into the deep depression on the surface. A similar physical mechanism for stability is known in the Hele-Shaw and other problems [11]. The closer the perturbation is to the tip of the finger initially, the longer it will take it to drift along the side of the finger and disappear. Fortunately, the nonvanishing viscosity prevents us from localizing a perturbation exactly at the tip as the perturbation must have a width of order ν or greater if it is to be unstable, so eventually all short-wavelength perturbations will be destroyed.

Now, let us proceed to the exact analysis. Let $F_0(x)$ be a stationary state, and consider a small perturbation $f(x)$. We will consider the evolution of the system in time, to linear order in f. One possibility to do this is, of course, to take the equation of motion (4), writing $F(x)=F_0(x)+f(x)$, and directly derive the equation of motion to linear order in f, obtaining $\partial_t f(x,t)=Lf$, where L is some linear operator whose eigenvalues describe the stability of the state F_0. However, this procedure would be very awkward, and we will use a different technique.

Let $F_0(x)$ be a state formed using N poles, at positions $\epsilon_1,\epsilon_2,\ldots,\epsilon_N$. We will write a state $F(j)$, near to $F_0(j)$, as

$$F(j)=F_0(j)+\frac{f(j)}{j}+i\sum_{a=1}^{N}\nu f_a e^{-i\epsilon_a j} \tag{10}$$

where $f(j)$ is a function of j and the various f_a are numbers representing small deviations of the pole coordinates from their original positions. By doing this, we have introduced some redundancy in describing the possible perturbations to F_0.

If $f(j)=0$, then we can derive a linear equation of motion for the f_a, simply using the pole dynamics equation. We must place the N poles at positions $\epsilon_1+f_1,\epsilon_2+f_2,\ldots,\epsilon_N+f_N$ and linearize Eq. (7) about $f_a=0$. We will obtain some equation of the form

$$\partial_t f_a=L_{a,b}f_b+M_{a,b}\overline{f_b} \tag{11}$$

for some linear operators L,M, with an implied sum over b. The result of this is that there are $2N$ different modes (there are N complex coordinates f_a). One of these modes is a zero mode; this is a mode with all f_a equal to the same real constant. From the analysis above, if all ϵ_i have the same real coordinate, then the other $2N-1$ modes are all stable (have negative eigenvalue).

Now, consider the case with nonzero $f(j)$. For simplicity, first consider a case with initial conditions such that $f(1)$ is nonzero, but all other $f(j)$ are zero. Then, the linearized equation of motion yields

$$\partial_t[F(j)-F_0(j)]=(1-\nu 1^2)\delta_{j,1}f(1)+2\nu\times\sum_{i=1}^{N}[1-\delta_{j,1})f(1)e^{-i\epsilon_i(j-1)}-\overline{f(1)e^{-i\epsilon_i(j+1)}}]. \tag{12}$$

The reason for the factor of $(1-\delta_{j,1})$ in the sum is that in Eq. (6) each pole contributes only to $F(k)$ for $k>0$. It is convenient now to rewrite Eq. (12) as

$$\partial_t[F(j)-F_0(j)]=(1-\nu 1^2-2\nu N)\delta_{j,1}f(1)+2\nu\sum_{i=1}^{N}[f(1)e^{-i\epsilon_i(j-1)}-\overline{f(1)e^{-i\epsilon_i(j+1)}}], \tag{13}$$

which, combined with Eq. (11), is equivalent to the equations

$$\partial_t f(1)=(1-\nu 1^2-2\nu N)f(1), \tag{14}$$

$$\partial_t f_a=-2i[f(1)e^{i\epsilon_a}-\overline{f(1)e^{-i\epsilon_a}}]+L_{a,b}f_b+M_{a,b}\overline{f_b}. \tag{15}$$

If we extend this procedure to the case of general $f(j)$, we find

$$\partial_t f(j)=(j-\nu j^2-2j\nu N)f(j)+2\nu\sum_{k>j}\sum_{a=1}^{N}(e^{i(k-j)\overline{\epsilon_a}}-e^{+i(k-j)\epsilon_a})f(k), \tag{16}$$

$$\partial_t f_a=-2i\sum_j[f(j)e^{ij\epsilon_a}-\overline{f(j)e^{-ij\epsilon_a}}]+L_{a,b}f_b+M_{a,b}\overline{f_b}. \tag{17}$$

The above two equations fully define the linear evolution of the system. Note that the matrix describing the linear evolution of the system is *triangular*: We find that $\partial_t f(j)$ depends only on $f(k)$ for $k\geqslant j$. This makes it possible to directly read off the eigenvalues of the matrix. There are $2N$ eigenvalues which are just the eigenvalues from the evolution of Eq. (11). Then there are eigenvalues which are $j(1-2\nu N)-\nu j^2$ with $j=1,2,3,\ldots$. Each of these eigenvalues must in fact be counted twice, since there is one such eigenvalue with $f(j)$ purely real, and one such with $f(j)$ purely imaginary. So long as we consider the stationary state found above, with N poles, all with the same imaginary coordinate and with N the largest integer less than $(1+\nu)/2\nu$, then all eigenvalues of the linear evolution are negative, with one (or three) exceptions. The first exception is the zero mode corresponding to changing the imaginary coordinate of all the poles by the same amount. The other two possible exceptions occur if $(1+\nu)/2\nu$ is an integer, in which case $1-2\nu N-\nu 1^2=0$, and so we have two more zero modes. However, in any case, there are no positive eigenvalues.

This result confirms the numerical result for the eigenvalues [12]. In that work, an approximate argument was given which led to the same series of eigenvalues based on constructing linear perturbations of the system by adding poles. We have been able to demonstrate these eigenvalues exactly by reducing the matrix to triangular form, which also enables a full calculation of all eigenvectors. For an eigenvalue $j(1-2\nu N)-\nu j^2$, the eigenvector is $f(j)=1,\ f(j-1)=[\nu/(1-\nu N-\nu j)]\Sigma_{a=1}^{N}(e^{i\epsilon_a}-e^{+i\epsilon_a}),\ldots$. Further, we have shown that the off-diagonal elements of the stability matrix are exponentially large. This is connected to the fact that, in the WKB analysis, perturbations will grow exponentially before they reach $x=0$. This indicates that, despite the linear stability, an exponentially small perturbation can lead to nonlinear instability.

The above analysis might be slightly confusing, as it seems we have introduced more eigenvectors than we started with by adding the f_a coordinates. We now show how to correctly count eigenvectors to resolve this. If we look at the linearized equation of motion in its original form, without introducing the additional coordinates f_a, we notice that the diagonal term in the linear equation of motion is $j-\nu j^2$. For very large j, this term must dominate all other terms, and so, if we order the eigenvalues of the linearized equation of motion, and look at the $2j$th eigenvalue, this eigenvalue must be close to $j-\nu j^2$. This gives us one way to count the number of eigenvalues of the problem: since the dimension of the space of $f(j)$ is infinite, we cannot simply count the number of eigenvalues directly, but we can count the number of eigenvalues less than a given number. Now, notice that

$$j(1-2\nu N)-\nu j^2=k-\nu k^2-N+N^2\nu \tag{18}$$

with

$$k=j+N. \tag{19}$$

The quantity $-N+N^2\nu$ will be of order N, but for very large k it will be small compared to $k-\nu k^2$. So, while we have introduced $2N$ eigenvalues by adding the f_a coordinates, Eq. (19) makes it clear that we have made up for this by losing $2N$ eigenvalues elsewhere: the full set of eigenvalues includes the $2N$ eigenvalues from Eq. (11) as well as eigenvalues $j-\nu j^2-N+N^2\nu$ with $j=N+1,N+2,N+3,\ldots$. Then, if we look at the $2j$th eigenvalue, it will be close to $j-\nu j^2$. Thus, we have found all eigenvalues of the linear evolution. This counting of eigenvalues is reminiscent of the problem of anomalies in field theory.

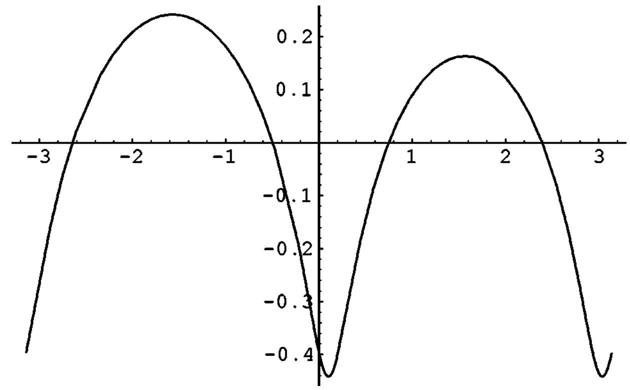

FIG. 1. Competition between two fingers, short times. Poles have real coordinates 0.1 and $\pi-0.1$.

In the limit of vanishing viscosity, the poles form a fluid, the density of which in the fingering solution can be calculated. In this case, the eigenvalues of the stability matrix extend down to zero. One finds that [9] $h(x)$ has a logarithmic divergence in its derivative as $x\rightarrow 0$. This contrasts with the case at $\eta=1$, for which the stable finger occupies only half the channel. We expect that for small $0<\eta<1$ the width of the finger compared to the channel width is intermediate between 1/2 and 1.

COMPARISON TO BRANCHED GROWTH MODEL

Let us consider the relation between the pole dynamics and the branched growth model dynamics [6]. The dynamics are similar, with one complication that highlights the problems in representing an evolving surface with a finite sum of poles.

Consider first a situation with a collection of N poles located at $x=0$, such that N is close to the maximal number of poles, describing a single finger. Add to this some number of poles at $x=\pi+\delta$, with δ small. This describes a split in the tip, leading to two branches. The solution is unstable, and one finds that δ grows exponentially in time, with a time constant that is independent of ν. Thus, one of the two branches will win, and the other branch will disappear. As a simple example, we show a situation with only two poles,

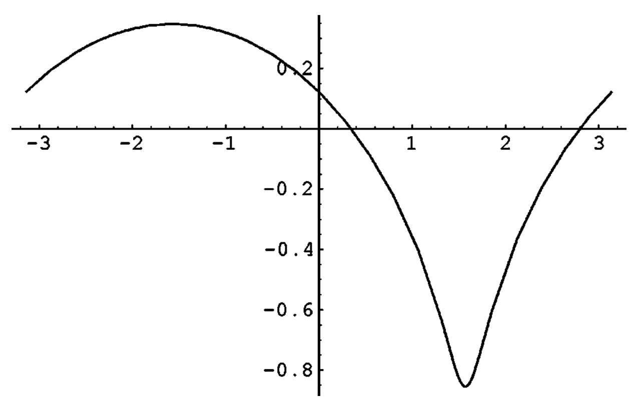

FIG. 2. Competition between two fingers, long time limit where one finger is completely suppressed.

one located at $x=0.1$ and the other at $x=\pi-0.1$. The two branches are initially almost symmetric, as shown in Fig. 1. Then, at long times, in Fig. 2, we show the final configuration with only one branch. The poles describe the points separating two branches.

While the dynamics is thus qualitatively similar to that of Halsey, there is a complication. Consider now a situation with the maximal number of poles, but let the poles all be at different x, such that the poles are separated by distances of order ν. This describes a situation with a large number of branches. As the real positions of the poles merge, some branches die while others become larger. This process is physically exactly what is expected in branched growth dynamics; however, in the pole dynamics, we lack exact solutions of the pole equations of motion for such a case. Thus, we are so far unable to describe the formation of larger branches using the poles. This is a problem for future work.

CONCLUSION

In conclusion, we have considered the problem of the dielectric breakdown model in the limit of vanishing η. This leads to a pole dynamics, which enabled us to find the exact eigenvalues of the linear stability matrix. Going from $\eta=0^+$ to nonvanishing η is likely to be a very nontrivial step. For $\eta=0^+$ DBM clusters are two-dimensional, with a one-dimensional surface. For nonvanishing η, while the cluster dimension may be only slightly reduced from 2, the surface dimension jumps and becomes equal to the cluster dimension. Thus, the transition for small η may be highly singular [13].

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